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Positioning and Pricing of Horizontally Differentiated Products

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We provide structural results and a solution method for designing horizontally differentiated product lines – optimizing product positions and prices – under fairly general consumer choice behavior. Our choice model is a generalization of the basic Hotelling-Lancaster locational choice model: Consumer tastes (ideal products) follow a general distribution; substitution disutility (transportation cost) can be an asymmetric convex function of product-spatial distance; and the market may not be fully covered. We formalize the notion that a shift of consumer tastes toward one end of the product space cannot result in a shift of the optimal product line in the opposite direction. For a unimodal taste distribution, we show that with respect to the product that covers the mode (or one of two products adjacent to it) prices *and* market shares drop toward the tails. Hence, higher popularity is always associated with higher price – although pricing, positioning, relative market share and popularity of products are all endogenous in our model. Our solution method is exact for discrete consumer taste distributions. Whereas, for continuous distributions, it requires lower and upper bounds, which can be computed efficiently using shortest path formulations and they asymptotically converge to the optimal profit.

Key words: product variety management; assortment planning; retail operations; horizontal differentiation

1. Introduction

Consumer preferences vary on matters of taste. Someone's ideal color for a night dress may be another person's nightmare. Faced with heterogeneity in tastes, firms offer a menu of products differentiated by taste attributes, a practice called *horizontal product differentiation*. Composing horizontally differentiated product lines, with each product targeting a certain range of tastes, involves product positioning and pricing decisions that collectively influence consumer choice behavior.

In practice we often see firms offering product lines that exhibit pure horizontal differentiation. For example, most electronic gadgets come in several fashionable colors, chocolate bars differ in cacao content (37%, 59%, 74% and 87%) and cheddar cheese has varying sharpness of flavor depending on its age (1, 2, 4, 6 and 10 years). Other examples include: color in furniture, apparel and countless other fashion items; and flavor, consistency, spiciness, fattiness, etc. of food products (orange juice with no pulp, some pulp or lots of pulp; mild, medium or spicy salsa dipping sauce; whole, reduced or no fat yogurt). Horizontally differentiated variants of the same product usually have the same price (as in orange juice, salsa and yogurt) but we also see different prices being charged (for cheese, some chocolate bars, electronic gadgets and furniture items).

Product line design for horizontal differentiation must take into account how consumer tastes are distributed and how willing consumers are to substitute away from the product they prefer most. In this paper, we study how consumer preferences shape the optimal product line design – involving product positioning and pricing decisions – aimed at achieving horizontal product differentiation. We use the Hotelling-Lancaster locational choice model under fairly general assumptions: consumer tastes follow a general distribution; the disutility of substituting away from an ideal product has a general form (an asymmetric convex function of product-spatial distance) and the firm does not have to cater to all the consumers (less than full market coverage is allowed).

There is no known solution to the product line design problem with such general assumptions; it is common practice in the literature – for tractability reasons – to assume a uniform consumer taste distribution, symmetric and linear transportation costs, and complete market coverage (e.g., de Groote 1994, Balasubramanian 1998, Dewan et al. 2000). In contrast, we start by establishing structural results on the optimal product line design under these general assumptions and then develop a solution method using these structural results. Among our structural results, we show that a unimodal consumer taste distribution requires products with higher price *and* higher market share be placed where the consumer taste distribution peaks. In consequence, prices and market shares in the optimal assortment are unimodal, implying that products with higher popularity should be more expensive. Note that product positions, prices and market shares, hence the popularity of products, are endogenous in our model. We also derive a novel and very intuitive monotonicity result on the optimal assortment: if the consumer tastes shift such that they favor one end of the product space, the optimal assortment cannot move toward the opposite end. Finally, we develop a shortest path formulation of the product line design problem by discretizing the product space, and demonstrate that it produces a near-optimal solution that asymptotically converges to the optimum. We accomplish this by developing effective lower and upper bounds on the true optimal profit and demonstrating that optimality gaps lower than 1% can be achieved within seconds.

Locational choice models have been used extensively in operations management research to model consumer behavior. de Groote (1994) focuses on operations-marketing coordination issues around how a firm's breadth of product line gets decided. Chen et al. (1998) explore optimal product line design under full market coverage when consumers have heterogenous reservation prices. Gaur and Honhon (2006) solve an assortment-inventory problem in static and dynamic substitution settings with exogenously set prices and study the value of considering dynamic substitution in assortment planning with a unimodal consumer taste distribution. Jiang et al. (2006) compare optimal product positioning policies under two settings: offering a set of standard products that cannot be altered or a set of base products that can be customized to enable downward substitution. They study the properties of optimal assortments for monotone consumer taste distributions when the firm can engage in first-degree price discrimination for custom products (hence pricing is not a decision variable). Alptekinoğlu and Corbett (2010) study inventory replenishment and product line design decisions of a firm that can offer custom products that are made to order and standard products that are made to stock to achieve a shorter delivery leadtime. They consider a general distribution of consumer tastes and assume a linear and symmetric transportation cost as well as full market coverage. Their structural results focus on the mix of standard and custom products in the optimal product line design. Honhon et al. (2012) develop methods to find the optimal assortment under exogenous prices and a ranking-based choice model that includes the locational choice model with fixed product locations and prices as a special case.

Logit choice models have also been popular in addressing assortment planning problems in operations management, due in large part to their convenient analytical properties. Papers that consider pricing and assortment selection issues using the basic multinomial logit (MNL) or nested logit choice models include Cachon and Kök (2007), Aydın and Porteus (2008), Rusmevichientong et al. (2010) and Li and Huh (2011). We defer to Gaur and Honhon (2006) for a comparison between the two choice models and to Li and Huh (2011) for a recent review of logit based literature.

Our contribution is two-fold: (i) we establish structural results on the optimal assortment and prices under general conditions (continuous nonuniform distributions, nonlinear and asymmetric transportation costs and the possibility of less than full market coverage); and (ii) we develop and study the performance of various discretization methods that make the problem amenable to a shortest path formulation and provide near-optimal solutions. We hope that these results will be of value for assortment planning in multiperiod, dynamic or competitive settings. Our discretization methods can be helpful in achieving computational tractability in these problems, which often suffer from the curse of dimensionality.

In the next two sections we formulate the product line design problem and present our structural results. We then develop our solution method in §4 and study its performance numerically in §5. All proofs are presented in Appendix A.

2. Product Line Design Problem

We consider consumers making a choice among a set of horizontally differentiated products. Let $\Omega \subseteq \mathbb{R}$, a convex set of real numbers, represent the product space. Each product is characterized by a single taste attribute, an element of Ω, which we refer to as product's *location*. We assume that this attribute can be measured on a cardinal scale such as fat content for dairy products. Alternatively, product location can be interpreted as the ratio of two product attributes (Lancaster 1998), as long as unit cost does not vary with these attributes.

There is a firm operating in this product space that decides how many products to offer, where in the product space to locate its products, and what price to charge for each. Let n denote the number of products, $x_i \in \Omega$ be the location and p_i be the price of product i. The firm must choose number of products n, product locations $\mathbf{x} \equiv (x_1, ..., x_n)$, and prices $\mathbf{p} \equiv (p_1, ..., p_n)$. The variable unit cost of production for a product with any location in Ω is $c \geq 0$ and there is a fixed cost $k > 0$ for offering each product.

The consumer population is heterogenous in their tastes: they each have a *location* in Ω that represents their *ideal product* - the product they would purchase if price was not a factor at all. While consumers know their own location, from the firm's perspective, the location of any given consumer is a continuous random variable Y with cumulative distribution function (cdf) F and probability density function (pdf) f.

A consumer with location y receives utility $U(y, x_i, p_i) = \bar{p} - p_i - d(x_i - y)$ from a product located at x_i and priced at p_i. Here, $\bar{p}$ is the consumer's willingness-to-pay for her ideal product, typically referred to as the *reservation price*. Also, consumers suffer an asymmetric and nonlinear disutility from having to consume a non-ideal product. We model this by a *transportation cost function*, $d(x_i - y)$, whose argument is the distance from the consumer's location to the product's location. Specifically, we assume that $d(\cdot)$ is a positive, finite-valued convex function that has a minimum at zero, $d(0) = 0$, and is differentiable everywhere else such that $0 < d(z) < \infty$, $|d'(z)| > 0$, and $d''(z) \geq 0$ for all $z \neq 0$. The reservation price, $\bar{p}$, and transportation cost function, d, are assumed to be the same for all consumers.

We let the transportation cost function, d, to be asymmetric around each consumer's ideal product to capture settings where positive and negative deviations from a consumer's ideal product result in different disutilities. For example, a consumer who ideally wants 2% milk might prefer 1% milk to 3% milk in the absence of her ideal product. Convexity of d reflects the possibility that larger deviations in either direction may cause higher disutility at an increasing rate.

Consumers have an outside option with a fixed utility, set to zero without loss of generality. Each consumer chooses to buy the product that gives her the highest utility, or opts out and obtains zero utility from not buying any of the n products. That is, a consumer located at y buys product i if

$i = \arg\max_{j\in\{1,2,\dots,n\}}\{\bar{p} - p_j - d(x_j - y)\}$, and $\bar{p} \geq p_i + d(x_i - y)$. If no such product exists, meaning that all n products result in negative utility, the consumer chooses the outside option.

This locational consumer choice model dates back to Lancaster (1966, 1975), who extended the work of Hotelling (1929) on spatial competition. The Hotelling-Lancaster model assumes a uniform density of consumers on a continuous attribute space. In contrast, we assume a general continuous distribution, and extend our work to discrete distributions in Appendix B.

We let *market size*, m, be the total number of consumers in the market and consider it as a deterministic parameter for simplicity (one can also let the market size be a random variable with expected value m as long as the market size is independent of consumer locations and firm decisions). We call an interval the *market segment* of a product, if for all consumers with locations within that interval, purchasing a unit of the product is a utility-maximizing choice among the n products offered by the firm and the outside option.

The firm's expected profit can then be expressed as

$$\pi(n,\mathbf{x},\mathbf{p}) = -kn + \sum_{i=1}^{n} m(p_i - c)P\{Y \in a_i(\mathbf{x},\mathbf{p})\}$$

where product i captures the market segment $a_i(\mathbf{x},\mathbf{p}) = [\underline{x}_i, \overline{x}_i] \subseteq \Omega$ and commands a *market share* of $P\{Y \in a_i(\mathbf{x},\mathbf{p})\} = F(\overline{x}_i) - F(\underline{x}_i)$, which is the probability that a consumer buys product i. (Henceforth we suppress the dependence of a_i on product locations $\mathbf{x}$ and prices $\mathbf{p}$.)

To sum up, the firm's product line design problem is to decide the number and locations of products to offer $(n,\mathbf{x})$ and to set the prices of the products $(\mathbf{p})$ so as to maximize $\pi(n,\mathbf{x},\mathbf{p})$. Let $\pi^* = \max_{n,\mathbf{x},\mathbf{p}} \pi(n,\mathbf{x},\mathbf{p})$ denote the optimal expected profit; and n^*, $\mathbf{x}^*$ and $\mathbf{p}^*$ be the optimal number of products, and the optimal product location and price vectors, respectively. Throughout, we assume $c < \overline{p}$ so that the market is profitable.

3. Structural Results on the Optimal Assortment

In this section, we explore the properties of the optimal assortment to gain insights into which products the firm should offer in the assortment and how it should price them. Because there is a fixed cost associated with every product included in the assortment, the firm cannot just offer dedicated products for each consumer and charge the reservation price. Instead, the firm needs to offer an assortment that can attract consumers whose ideal products do not necessarily match any of the products offered. The firm can ensure this by charging a price lower than the reservation price. The larger the range of locations the firm wants to cover with a single product, the lower the price it should charge for that product. We refer to the difference between the reservation price and the price charged for a product as the *price differential*.

We first show a basic structural property of the optimal assortment. This property helps us derive more structural results in this section and is also essential to formulate the product assortment problem as a dynamic programming (DP) problem in the next section and solve it.

LEMMA 1. *Optimal product locations and prices, $\mathbf{x}^*$ and $\mathbf{p}^*$, must satisfy the following properties:*

(a) Market segments that result from $\mathbf{x}^*$ and $\mathbf{p}^*$ are closed intervals with boundary points $\underline{x}_i^*, \overline{x}_i^* \in \Omega$ such that $\underline{x}_i^* < \overline{x}_i^*$ for $i = 1, \ldots, n$ and $\overline{x}_i^* \leq \underline{x}_{i+1}^*$ for $i = 1, \ldots, n-1$.

(b) Products are located and priced such that consumers at the boundary points of market segments obtain zero utility from purchase. That is, with a market segment of length $\xi_i^* = \overline{x}_i^* - \underline{x}_i^*$, product i is located at $x_i^* = \underline{x}_i^* + L(\xi_i^*)$ and priced at $p_i^* = \bar{p} - T(\xi_i^*)$ for $i = 1, \ldots, n$, where $L(\xi_i^*)$ is the distance from the lower boundary point to the product location, $T(\xi_i^*)$ is the price differential, and they satisfy $d(L(\xi_i^*)) = d(L(\xi_i^*) - \xi_i^*) = T(\xi_i^*)$.

Hence, the optimal product line design must have non-overlapping market segments in the sense that no consumer obtains strictly positive utility from more than one product, and that consumers at market segment boundary points obtain zero utility. See Chen et al. (1998) and Alptekinoğlu and Corbett (2010) for similar results in different contexts.

The most important implication of Lemma 1 is that the product line design problem can be reformulated in terms of market segment boundary points rather than product locations and prices. That is, optimizing the product locations $\mathbf{x}$ and prices $\mathbf{p}$ is equivalent to optimizing $\mathbf{a} \equiv (a_1, \ldots, a_n) = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$, which we refer to as the *assortment*. Once the firm chooses the

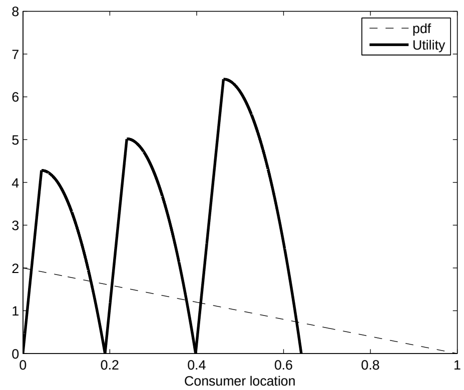


Figure 1 Optimal assortment in Example 1.

boundary points $[\underline{x}_i, \overline{x}_i]$ optimally for each product i, with n also considered as a decision variable, the optimal product locations and prices are also effectively set. Given a market segment of length $\xi_i = \overline{x}_i - \underline{x}_i$, because consumers at the boundary points must have 0 utility, the transportation cost incurred by those consumers must be the same and the price differential must be equal to it, i.e., $d(L(\xi_i)) = d(L(\xi_i) - \xi_i) = T(\xi_i)$. That is, the firm has to lower the price just enough to compensate for the transportation cost that the consumers at the boundary points incur. All other consumers within the market segment receive a strictly positive utility.

We now provide an example where we calculate the price differential and product locations.

Example 1. Let $\bar{p} = 25$, $c = 5$, $m = 1$, $k = 2$, and $d(z) = 100z$ if $z \geq 0$, $200z^2$ if $z < 0$. Assume that the consumer tastes follow a Beta distribution with parameters $\alpha = 1$ and $\beta = 2$, i.e., $f(y) = \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}$ for $y \in (0,1)$ where B is the Beta function. The optimal assortment is $\mathbf{a}^* = ([0, 0.1891], [0.1891, 0.3977], [0.3977, 0.6409])$ with market segment lengths $\xi_1 = 0.1891$, $\xi_2 = 0.2086$ and $\xi_3 = 0.2432$. We defer the discussion of how to find the optimal solution to §4. See Figure 1 for a depiction of the consumer taste distribution (dashed line) and the utility derived by consumers who purchase each product in the optimal assortment (solid lines). Zero-utility points are the market segment boundary points. The product that covers a given market segment is located where the utility is maximized; the consumer who happens to have that ideal product does not incur any transportation cost and therefore gets the highest utility within that segment. Because the utility for that consumer is $\bar{p} - p_i(a_i)$, the price of a product is inversely proportional to the maximum height of the utility curve in each market segment. Solving for $d(L(0.1891)) = d(L(0.1891) - 0.1891)$, we get $L(0.1891) = 0.0428$. Similarly, we find that $L(0.2086) = 0.0502$ and $L(0.2432) = 0.0641$. Therefore, products are located at $x_1^* = 0 + 0.0428 = 0.0428$, $x_2^* = 0.1891 + 0.0502 = 0.2393$ and $x_3^* = 0.3977 + 0.0641 = 0.4618$. Also, we have the price differentials $T(0.1891) = 100 \times 0.0428 = 4.28$, $T(0.2086) = 5.02$ and $T(0.2432) = 6.41$; and finally the optimal prices $p_1^* = 25 - 4.28 = 20.72$, $p_2^* = 25 - 5.02 = 19.98$ and $p_3^* = 25 - 6.41 = 18.59$.

Letting $\mathcal{A}$ be the set of all possible assortments that satisfy the properties shown in Lemma 1, we rewrite the firm's product line design (or assortment planning) problem as follows:

$$\pi^* = \max_{\mathbf{a}\in\mathcal{A}} \pi(\mathbf{a}) = -kn + \sum_{i=1}^{n} m[p_i(a_i) - c]P\{Y \in a_i\},$$

where $p_i(a_i) = \bar{p} - T(\overline{x}_i - \underline{x}_i)$ and $P\{Y \in a_i\} = F(\overline{x}_i) - F(\underline{x}_i)$. Note that it is possible for the firm not to cover the entire market, that is, we can have $\sum_{i=1}^{n} P\{Y \in a_i\} < 1$.

Next, we study how the market shares and prices compare in the optimal assortment. Where should the firm locate the most expensive products? Which products should have the highest market share? We first prove two general results and then focus on the unimodal case.

Lemma 2. *The price differential $T(\xi)$ is increasing and convex in market segment length ξ.*

Naturally, if the firm wants to capture a wider market segment with a single product, it has to decrease the price. Furthermore, as the market segment length increases, the rate of price decrease required increases.

Our next result, which uses Lemma 2, relates the prices and market shares of two adjacent products in the optimal assortment. It implies that a higher market share is always coupled with a higher price; one of two adjacent products will not only command a higher price but also a higher market share (a higher purchase probability). Note that a higher price necessarily means a narrower market segment - and vice versa - due to Lemmas 1b and 2.

Lemma 3. *Suppose the optimal assortment has two products that cover adjacent segments: $[a,x]$ and $[x,b]$. Let e be the point that divides $[a,b]$ into two equiprobable intervals, i.e., $F(e) - F(a) = F(b) - F(e)$. Then, the common boundary point x must be between e and $(a+b)/2$. That is, we must have $x \in [e,(a+b)/2]$ if $e \leq (a+b)/2$, or $x \in [(a+b)/2, e]$ if $e \geq (a+b)/2$.*

A direct implication of the structural results above is that if the consumer taste distribution is uniform on Ω, then products in the optimal assortment have equal market segment lengths, identical prices and market shares. For a general consumer taste distribution, F, however, the optimal assortment is not as simple.

3.1. Unimodal Tastes

Although a local result about any two adjacent products in the optimal assortment, Lemma 3 suggests that more expensive products will tend to be located in higher-density locations where they can attract a higher market share. There is a clean implication of this general observation in the case of unimodal taste distributions: the optimal prices and market shares have to be unimodal.

THEOREM 1. *If the pdf of consumer tastes, f, is unimodal, then the optimal assortment $\mathbf{a}^*$ must satisfy the following properties:*

(a) The mode M, defined as the maximizer of f, must be covered.

(b) All covered market segments must be adjacent to each other, i.e., $\overline{x}_i^* = \underline{x}_{i+1}^*$ for $i = 1, \ldots, n-1$.

(c) Prices and market shares must be unimodal. That is, there must exist a product i_0 such that it has the highest price and the highest market share, and going from product 1 to i_0 (from product i_0 to n^*) prices and market shares must both increase (decrease). Furthermore, i_0 must be either the product that covers the mode M or one of the two adjacent products. If f is symmetric unimodal or monotone, then i_0 is the product that covers the mode.

Thus, the optimal assortment under a unimodal consumer taste distribution clusters products around the mode and leaves non-covered consumer locations (if any) at the tails. Also, generally speaking, prices and market shares drop as one moves away from the mode toward either of the tails. Because prices and market shares are unimodal and they move together, more popular products are always more expensive. Note that the positions, prices, relative market shares, and popularity of products are all endogenous in our model. In contrast, MNL choice model does not allow relative market shares to vary. Due to the so called Independence of Irrelevant Alternatives (IIA) property, relative market shares of any two products (the ratio of their choice probabilities) remain a constant regardless of which other products are included in the assortment. Also, popularity ordering of products is exogenous under MNL choice, because higher attractiveness implies higher market share independent of the assortment. In our model, product popularity is fully endogenous as it depends on which other products are introduced, at what locations and at what prices.

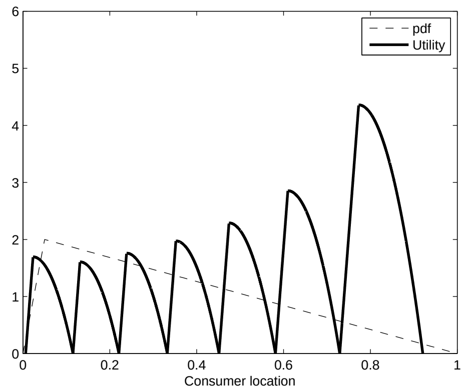


Figure 2 Optimal assortment in Example 2.

One would further expect the highest-priced product with the highest market share to always cover the mode. Theorem 1c shows that this is true for unimodal distributions that are symmetric or monotone. Example 2 demonstrates that this property does not necessarily hold for all unimodal distributions. The product with the highest market share and price can be one of the two products adjacent to the one that covers the mode.

Example 2. Consider a triangular consumer taste distribution on $[0,1]$ with a mode at 0.05 (shown in Figure 2 with dashed lines). Set $\bar{p}=25$, $c=5$, $m=1$, $k=0.5$, and $d(z)=100z$ if $z\geq 0$, $200z^2$ if $z<0$. The optimal solution is to offer the seven products shown in Figure 2 (please see Example 1 for a description of how to interpret the figure). Note that the first product covers the mode of the distribution but it neither has the highest price nor the highest market share (the first product's market share is 0.18, versus 0.19 for the second product). The second product has the highest price and the prices monotonically decrease going from the second to the seventh product.

Jiang et al. (2006) have a related result in a specialized setting with downward customization (a base product being altered to match the ideal products of lower locations), first-degree price discrimination and linear transportation costs. They prove that the market shares of base products are increasing and the distances between them are decreasing as the density of consumer tastes increases (Proposition 4, p. 30). Our result in Theorem 1, more general in many ways (e.g., unimodal distribution, nonlinear transportation costs, pricing decisions), complements this result. Another related result is by Alptekinoğlu and Corbett (2010): for a unimodal consumer taste distribution, they find that it is optimal to locate standard products around the mode and custom products at the tails, a result driven by leadtime-variety tradeoff between standard and custom products.

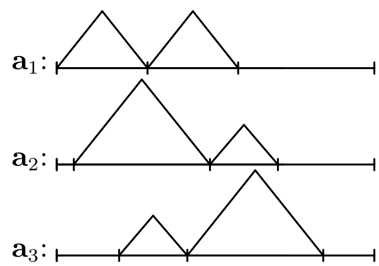


Figure 3 Definition 1 imposes a partial order: $\mathbf{a}_2$ and $\mathbf{a}_3$ are to the right of $\mathbf{a}_1$; $\mathbf{a}_2$ and $\mathbf{a}_3$ cannot be ordered.

3.2. Shifting Tastes

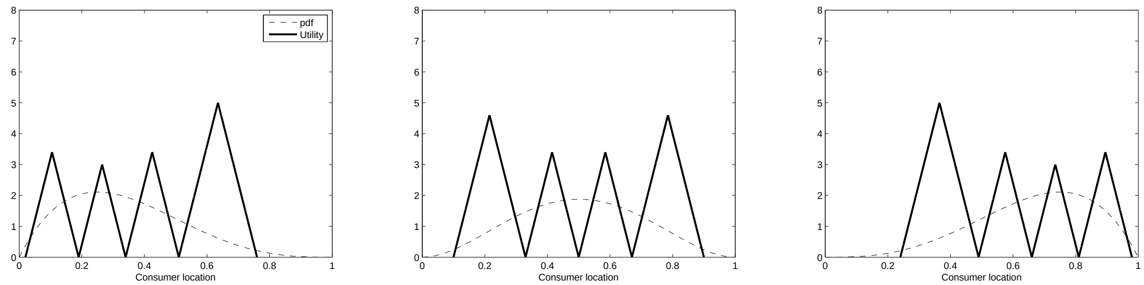
In this section, we study how the optimal assortment changes as the distribution of consumer tastes shifts. Suppose the product space is $[0,1]$, the consumer taste distribution is uniform on $[0,0.5]$ and the transportation cost function is symmetric linear, i.e., $d(z) = d|z|$. For small enough values of d and large enough values of k, the optimal assortment is to place a product exactly at 0.25 which covers the interval $[0,0.5]$. If instead, f is uniform on $[0.5,1]$, then the optimal assortment shifts 'to the right' and it becomes optimal to place a product at 0.75 covering the interval $[0.5,1]$. In this example, transferring probability from locations on the left of the product space to locations on the right results in the optimal assortment moving to the right.

To formalize this monotonicity property of the optimal assortment, we need to define a partial order on assortments that corresponds to *being to the right* and a stochastic order in probability distributions that represents the notion of *moving probability to the right*.

Definition 1. An assortment $\mathbf{a}' = ([\underline{x}'_1, \overline{x}'_1], \ldots, [\underline{x}'_n, \overline{x}'_n])$ is 'to the right' of assortment $\mathbf{a} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$, or $\mathbf{a}' \succeq_R \mathbf{a}$, if for any product i, we have $\underline{x}_i \leq \underline{x}'_i$ and $\overline{x}_i \leq \overline{x}'_i$. If $\mathbf{a}' \succeq_R \mathbf{a}$, we also say $\mathbf{a}$ is 'to the left' of $\mathbf{a}'$.

This definition partially orders assortments that have the same number of products. Take, for example, the assortments depicted in Figure 3; while both $\mathbf{a}_2$ and $\mathbf{a}_3$ are to the right of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ cannot be ordered using Definition 1.

We next define likelihood ratio (LR) dominance for probability distributions, which represents the notion of moving probability mass from consumer locations on the left to locations on the right. This stochastic order implies the more frequently used first order stochastic dominance (FOSD).



(a) Beta(2,4) (b) Beta(3,3) (c) Beta(4,2)

Figure 4 An example of optimal assortments shifting to the right ($\bar{p}=25$, $c=5$, $k=2$, $m=1$ and $d(z)=5|z|$).

DEFINITION 2. G LR-dominates F, or $G \succeq_{LR} F$, if $\frac{g(x)}{f(x)}$ is non-decreasing in x.

Our main result using LR dominance is that the optimal assortment cannot move to the left if the consumer taste distribution shifts to the right. See Figure 4 for an example with Beta distributions (Beta(4,2) $\succeq_{LR}$ Beta(3,3) $\succeq_{LR}$ Beta(2,4)) that shows how the optimal assortment changes.

THEOREM 2. *Suppose assortment* $\mathbf{a}^*$ *is optimal for consumer taste distribution* F*. Then, for any assortment* $\mathbf{a}$ *such that* $\mathbf{a}^* \succeq_R \mathbf{a}$*, and for any consumer taste distribution* G *such that* $G \succeq_{LR} F$*, we have* $\pi(\mathbf{a}^*; G) \geq \pi(\mathbf{a}; G)$*. Therefore, the optimal assortment cannot move to the left when the consumer taste distribution shifts to the right.*

This theorem provides sufficient conditions for a very intuitive monotonicity property to hold: if consumer tastes shift toward one end of the product space, the optimal assortment will not shift products toward the other end. The challenge is in specifying conditions under which this property holds. For example, LR-dominance is necessary to state our monotonicity result. In Appendix C we provide an example that shows Theorem 2 does not necessarily hold under an FOSD shift in the consumer taste distribution. Also, we remark that the profit function in this problem does not satisfy the properties generally used to show monotonicity of optimal decisions, such as increasing or single-crossing differences. We refer to Appendix C for further discussion.

4. Finding the Optimal Assortment

In this section we provide a set of tools for finding the optimal assortment. We consider a continuous taste distribution F with finite support $[z_L, z_H] \subseteq \Omega$, and set $z_L = 0$, $z_H = 1$ and $\Omega = [0, 1]$ to simplify the exposition. We first pose the problem as a DP.

Suppose the firm wants to solve the optimal product line problem for the interval $[z_t, 1]$. This problem can be written as a DP recursion where the value function $J(z_t)$ is the optimal expected profit from interval $[z_t, 1]$. Given a location z_t, the firm finds the next location z_{t+1} that maximizes the sum of expected profits from intervals $[z_t, z_{t+1}]$ and $[z_{t+1}, 1]$. The firm either covers the interval $[z_t, z_{t+1}]$ with a product – if the expected profit from that interval is positive – or chooses not to cover it. We take the terminal profit to be $J(1) = 0$. For any $0 \le z_t < 1$, we have

$$J(z_t) = \max_{z_t < z_{t+1} \le 1} \{\max\{0, [\bar{p} - T(z_{t+1} - z_t) - c]\,[F(z_{t+1}) - F(z_t)]\,m - k\} + J(z_{t+1})\} \tag{1}$$

To solve the product design problem, we need to solve $J(0)$, that is $J(0) = \pi^*$.

The expected profit function for each location z_{t+1} (the expression inside the first maximization statement) is generally not concave in z_{t+1} and may in fact contain multiple local maxima; there appears to be no efficient way of finding the optimal value of z_{t+1} at stage t of the DP. Rather than attempting to find the optimal solution, we propose to tackle the assortment problem in (1) by computing a lower bound via a discretization of the product space. We show that this method tends to the optimal solution as the discretization becomes finer. We also develop an upper bound, which allows us to evaluate the performance of the lower bound.

As an aside, if the consumer taste distribution is discrete, we show that the market segments of products in the optimal assortment must have boundary points at the locations with positive probability mass. This observation implies that the firm has to choose from finitely many possible assortments and that the exact optimal solution can be obtained easily (see Appendix B for details).

4.1. Obtaining a Lower Bound by Discretizing the Product Space

To find a lower bound on π^*, we impose a constraint that all market segment boundary points belong to a finite, discrete set of locations in Ω. This approach can be used with any discrete set and lends itself to an easily-solvable shortest path formulation. We first formally describe the technique, and then suggest four methods for selecting this set.

Let $\hat{Z} = \{\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_N\}$ be a set of $N+1$ locations in Ω such that $0 = \hat{z}_0 < \hat{z}_1 < \cdots < \hat{z}_N = 1$. We impose the following constraint on the problem: $\underline{x}_i, \overline{x}_i \in \hat{Z}$ for all products $i = 1, ..., n$ in assortment **a**. Let $J_{\hat{Z}}$ be the value function for this problem, and $J_{\hat{Z}}(1) = 0$ be the terminal profit.

The DP recursion then becomes:

$$J_{\hat{Z}}(z_t) = \max_{z_{t+1} \in \hat{Z},\ z_{t+1} > z_t} \{\max\{0, [\bar{p} - T(z_{t+1} - z_t) - c][F(z_{t+1}) - F(z_t)]m - k\} + J_{\hat{Z}}(z_{t+1})\} \qquad (2)$$

for $z_t \in \hat{Z}$ and $t = 0, 1, \ldots, N-1$. The firm's product line design problem is to solve $J_{\hat{Z}}(0)$. Because this problem has extra constraints, the profit obtained is a lower bound on the optimal profit π^*.

The solution to (2) can be found efficiently by casting the problem as a shortest path problem in an acyclic network. Let $V = \{0, 1, \ldots, N\}$ be a set of nodes, corresponding to consumer locations $\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_N$, respectively. Let $A = \{(v, v') : v, v' \in V \text{ and } v < v'\}$ be the set of arcs. We have $|A| = N(N+1)/2$. A unit flow on any arc $(v, v') \in A$ means that consumer locations $[\hat{z}_v, \hat{z}_{v'}]$ either make up a market segment, covered by one product with location $x_i = \hat{z}_v + L(\hat{z}_{v'} - \hat{z}_v)$ and price $p_i = \bar{p} - T(\hat{z}_{v'} - \hat{z}_v)$ (these follow from Lemma 1), or constitute a set of consumer locations that are not covered. For each arc $(v, v') \in A$, we define r as the profit contribution of the product if the segment is covered and zero otherwise:

$$r(v, v') = \max\{0, [\bar{p} - T(\hat{z}_{v'} - \hat{z}_v) - c][F(\hat{z}_{v'}) - F(\hat{z}_v)]m - k\}.$$

Finding the optimal assortment in the approximate problem is equivalent to finding the shortest path from node 0 to node N in the network described above, with arc lengths equal to $-r$. The worst-case computational complexity of the shortest path problem in an acyclic network is bounded by the number of arcs (see Ahuja et al. 1993, page 107). Therefore, our method of finding an approximate solution is of complexity $O(N^2)$. Note that a similar solution method is suggested by Chen et al. (1998) and used by Alptekinoğlu and Corbett (2010). One difference is that neither paper accounts for the possibility of not covering some portions of the market. In the example below, we solve for the best assortment for a given discretization using the shortest path formulation.

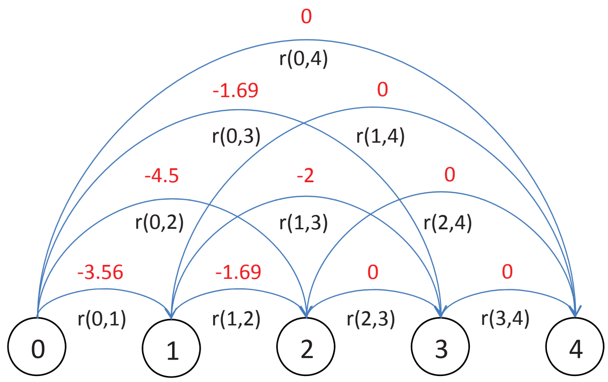


Figure 5 Network representation of the shortest path problem in Example 3.

Example 3. Let $\bar{p} = 25$, $c = 5$, $m = 1$, $k = 3$, and $d(z) = 40|z|$. Assume that the consumer taste distribution is a Beta(1,2). Let $\hat{Z} = \{0, 0.25, 0.5, 0.75, 1\}$. In this case $J_{\hat{Z}}(0) = 5.56$ and the assortment that yields the optimal profit is $\mathbf{a}^* = ([0, 0.25], [0.25, 0.75])$. Figure 5 depicts the network. The path that corresponds to assortment $\mathbf{a}^*$ goes from node 0 to node 1, then from node 1 to node 3 and finally from node 3 to node 4.

In the remainder of this section we suggest a number of ways to choose the set $\hat{Z}$. The first two are based on equidistant or equiprobable split of the product space. The next two are based on the minimization of a loss function. We prove that all four discretization methods converge to the optimal profit. We study their performance in the next section.

4.1.1. Equidistant and equiprobable discretization methods. The Equidistant Discretization (ED) method divides the product space into N intervals of equal length: $\hat{z}_i = \frac{i}{N}$ for $i = 0, 1, \ldots, N$, which imply $\hat{z}_i - \hat{z}_{i-1} = \frac{1}{N}$ for $i = 1, ..., N$. Let $\pi_D(N)$ be the lower bound on π^* obtained via the ED method with N equal-length intervals.

The Equiprobable Discretization (EP) method divides the product space into N intervals of equal probability: $\hat{z}_i = F^{-1}\left(\frac{i}{N}\right)$ for $i = 0, 1, \ldots, N$, which imply $F(\hat{z}_i) - F(\hat{z}_{i-1}) = \frac{1}{N}$ for $i = 1, ..., N$. Let $\pi_P(N)$ be the lower bound on π^* obtained via the EP method with N equiprobable intervals.

A finer discretization does not necessarily result in an improved lower bound: neither $\pi_D(N)$ nor $\pi_P(N)$ is generally monotone in N. In particular, it is possible to have $\pi_D(N) < \pi_D(N')$ or $\pi_P(N) < \pi_P(N')$ for $N > N'$, because the set $\hat{Z}^{N'}$ may happen to be closer to the optimal set of market segment boundary points than $\hat{Z}^N$. However, if we double the number of intervals by splitting each interval into two, then we get an improved lower bound. More formally, $\hat{Z}^N$ is always a subset of $\hat{Z}^{2N}$ and any feasible assortment to the problem with $\hat{Z}^N$ is also feasible for the problem with $\hat{Z}^{2N}$. This is true in general for any $j \in \mathbb{N}$ as stated in Lemma 4 (the proof is omitted).

Lemma 4. *$\pi_D(N) \le \pi_D(jN)$ and $\pi_P(N) \le \pi_P(jN)$ for all $j \in \mathbb{N}$.*

More importantly, we prove that these two lower bounds converge to the optimal solution.

Proposition 1. *The profits calculated using the ED and EP discretization methods converge to the optimal profit, i.e., $\lim_{N\to\infty} \pi_D(N) = \pi^*$ and $\lim_{N\to\infty} \pi_P(N) = \pi^*$.*

4.1.2. Loss function based discretization methods. In addition to using equidistant and equiprobable methods, we also study more sophisticated discretization methods that minimize expected loss from discretization as in Mease and Nair (2006). These methods try to find an optimal discretization $y(X)$ of a random variable X such that the expected loss, $\mathbb{E}\mathcal{L}[X - y(X)]$, is minimized. Because of the objective function, we expect these methods to have finer partitions in areas where the probability is high and coarser partitions in areas where the probability is low. This property may work to our advantage because areas with high probability are exactly the regions where the optimal assortment places products with narrow market segments (hence high prices) and high market shares (this follows from Lemma 3 and Theorem 1); finer partitions in such regions may yield solutions closer to the optimal assortment.

More formally, let $y(X)$ be an N-level discretized version of X such that $y(X)$ takes value $\hat{y}_i$ when X is in the interval $(\hat{z}_{i-1}, \hat{z}_i]$ for $i = 1, ..., N$. The objective is finding boundary points of the intervals, $\hat{z}_1, ..., \hat{z}_{N-1}$, and the points $\hat{y}_1, ..., \hat{y}_N$ at which to locate the mass in those intervals, that minimize $\mathbb{E}\mathcal{L}[X - y(X)]$ where $\mathcal{L}$ is a convex loss function such that $\mathcal{L}(x) = 0$ if and only if $x = 0$. Mease and Nair (2006) show that the optimal values of $\hat{z}_1, ..., \hat{z}_{N-1}$ and $\hat{y}_1, ..., \hat{y}_N$ can be obtained by iteratively solving the following two equations:

$$\hat{y}_i = \arg\min_u \mathbb{E}\mathcal{L}(X_i - u) \text{ for } i = 1, ..., N \tag{3}$$

$$\mathcal{L}(\hat{z}_i - \hat{y}_i) = \mathcal{L}(\hat{z}_i - \hat{y}_{i+1}) \tag{4}$$

where X_i is the random variable X truncated into the i-th interval $(\hat{z}_{i-1}, \hat{z}_i]$, $\hat{z}_0 = 0$ and $\hat{z}_N = 1$.

We use two very common loss functions: the squared loss function $\mathcal{L}(x) = x^2$ and the absolute loss function $\mathcal{L}(x) = |x|$ to guide partitioning of the product space Ω. For a given partition, $\hat{z}_1, ..., \hat{z}_{N-1}$, the squared loss (absolute loss) function requires placing the probability mass in each interval at the conditional mean (median); these are the optimal $\hat{y}_i$ points. Then, the boundary points, $\hat{z}_i$, given $\hat{y}_i$ need to be such that the boundary points are equidistant from adjacent $\hat{y}_i$'s. To start the iteration process, we use the $\hat{z}_i$ obtained with the ED method.

Note that we only need the partition on the product space, $\hat{z}_0, ..., \hat{z}_N$; the values of $\hat{y}_1, ..., \hat{y}_N$ are not useful in our setting. While we expect these methods to perform well, there is no guarantee of optimality as in Mease and Nair (2006), since our goal is not to minimize the expected loss function used to calculate the set $\hat{Z}$.

We refer to these two discretization methods as the SL (Squared Loss) and AL (Absolute Loss) methods, respectively. Let $\pi_S(N)$ and $\pi_A(N)$ be the lower bound obtained via discretizing the product space using the SL and AL methods, respectively. Unlike the ED and EP methods, we do not necessarily have $\pi_S(N) \leq \pi_S(jN)$ for $j \in \mathbb{N}$, because the set $\hat{Z}$ is generally not a subset of $\hat{Z}^{jN}$. However, we can prove that the two profit values do converge.

Proposition 2. *The profits calculated using the SL and AL discretization methods converge to the optimal profit, i.e.,* $\lim_{N\to\infty} \pi_S(N) = \pi^*$ *and* $\lim_{N\to\infty} \pi_A(N) = \pi^*$.

4.2. Obtaining an Upper Bound by Discretizing the Consumer Taste Distribution

To assess how good the lower bounds from the previous section are, we develop upper bounds on the optimal profit. We use the following basic idea: discretize the consumer taste distribution F in such a way that the optimal profit for this discrete distribution constitutes an upper bound on the optimal profit under F.

Given a continuous distribution F with support $\Omega = [0, 1]$, take an arbitrary set of locations $\tilde{Z} = \{\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_{N-1}, \tilde{z}_N\}$ such that $N \geq 2$, $\tilde{z}_1, \ldots, \tilde{z}_{N-1} \in \Omega$, and $0 = \tilde{z}_0 < \tilde{z}_1 < \cdots < \tilde{z}_{N-1} < \tilde{z}_N = 1$.

For each interval $[\tilde{z}_{j-1}, \tilde{z}_j]$, imagine taking all the probability mass in the interval, $g_j = F(\tilde{z}_j) - F(\tilde{z}_{j-1})$, and concentrating it at one point $\tilde{y}_j \in [\tilde{z}_{j-1}, \tilde{z}_j]$ within the interval. Let $G^{\tilde{Z}}$ be the discrete distribution that assigns probabilities $g_1, \ldots, g_N$ to points $\tilde{y}_1, \ldots, \tilde{y}_N$, respectively. Now the question is: How should one choose these $\tilde{y}$ points such that the optimal profit under $G^{\tilde{Z}}$ is guaranteed to exceed the optimal profit under F?

If tastes exhibit a symmetric unimodal or monotone distribution, obtaining an upper bound is fairly easy. We know that the optimal solution always has the most expensive product cover the mode (Theorem 1c). Hence, purchasing consumers who are closer to the mode pay a higher price. Concentrating all the probability mass in each interval at the point closest to the mode would then guarantee that the discrete problem would give a higher optimal profit than the optimal profit π^*.

PROPOSITION 3. *Suppose the pdf of consumer tastes, f, is symmetric unimodal or monotone with mode M. Let the m^{th} interval contain the mode, i.e., $M \in [\tilde{z}_{m-1}, \tilde{z}_m)$. Define $G^{\tilde{Z}}$ as the discrete distribution that assigns probability $g_i = F(\tilde{z}_i) - F(\tilde{z}_{i-1})$ to location $\tilde{y}_i$ for $i = 1, .., N$, where*

$$\tilde{y}_i = \begin{cases} \tilde{z}_i & \text{if } 1 \le i \le m-1, \\ M & \text{if } i = m, \\ \tilde{z}_{i-1} & \text{if } m+1 \le i \le N. \end{cases}$$

Solve the assortment problem under $G^{\tilde{Z}}$. The optimal profit obtained in this discrete problem, $\tilde{\pi}$, is an upper bound on the optimal profit π^*.

As we show in Appendix B, solving a discrete problem to optimality requires solving a shortest path problem very similar to the one posed in §4.1. Consequently, the worst-case computational complexity of obtaining an upper bound in symmetric unimodal and monotone cases is $O(N^2)$.

More generally, if we allow asymmetric unimodal distributions (that are not monotone), the same method fails to work. The reason is the possibility that the most expensive product may not be covering the mode; it can be one of the products adjacent to the product that covers the mode. We are no longer assured that consumers closer to the mode pay more. In order to handle this, we need to solve $N + 1$ discrete problems, which constitute an exhaustive search for at most one interval in which consumers located closer to the mode pay less in the optimal solution to the continuous problem (this follows from Theorem 1c). Discretizing the consumer taste distribution, F, as we describe below guarantees that every consumer pays a higher price in the solution to at least one of the $N+1$ discrete problems compared to the optimal solution, hence yielding an upper bound on π^*.

PROPOSITION 4. *Suppose the pdf of consumer tastes, f, is asymmetric unimodal with mode M. Let the m^{th} interval contain the mode, i.e., $M \in [\tilde{z}_{m-1}, \tilde{z}_m)$. Define $G_0^{\tilde{Z}}$ as the discrete distribution that assigns probability $g_i = F(\tilde{z}_i) - F(\tilde{z}_{i-1})$ to location $\tilde{y}_i$ for $i = 1, .., N$, where*

$$\tilde{y}_i = \begin{cases} \tilde{z}_i & \text{if } 1 \le i \le m-1, \\ \tilde{z}_{i-1} & \text{if } m \le i \le N. \end{cases}$$

Define $G_j^{\tilde{Z}}$ for $j = 1, .., N$ as the discrete distribution that makes the same probability assignments as $G_0^{\tilde{Z}}$ except that

$$\tilde{y}_j = \begin{cases} \tilde{z}_{j-1} & \text{if } 1 \le j \le m-1, \\ \tilde{z}_j & \text{if } m \le j \le N. \end{cases}$$

Solve the assortment problem under each one of these discrete distributions; let $\tilde{\pi}_j$ denote the optimal profit under $G_j^{\tilde{Z}}$ for $j = 0, 1, .., N$. The maximum profit obtained, $\tilde{\pi} \equiv \max_{j=0,1,..,N} \tilde{\pi}_j$, is an upper bound on the optimal profit π^*.

Given that the upper bound now requires solving $N+1$ shortest path problems, the worst-case computational complexity becomes $O(N^3)$. Therefore, the asymmetric unimodal case can be handled, but at the expense of more computational effort. Figure 6 presents two discretization examples with Beta distributions: one symmetric and one asymmetric.

Just like for the lower bound, $\tilde{Z}$ can be chosen using any of the discretization methods discussed in §4.1.1 and §4.1.2. Let $\tilde{\pi}_D(N), \tilde{\pi}_P(N), \tilde{\pi}_S(N), \tilde{\pi}_A(N)$ denote the value of the upper bound calculated when the product space is divided into N intervals according to the ED, EP, SL and AL methods, respectively. We show that, in all four cases, the upper bound also converges to π^*.

PROPOSITION 5. *We have $\tilde{\pi}_D(N) \to \pi^*$, $\tilde{\pi}_P(N) \to \pi^*$, $\tilde{\pi}_S(N) \to \pi^*$ and $\tilde{\pi}_L(N) \to \pi^*$ as $N \to \infty$.*

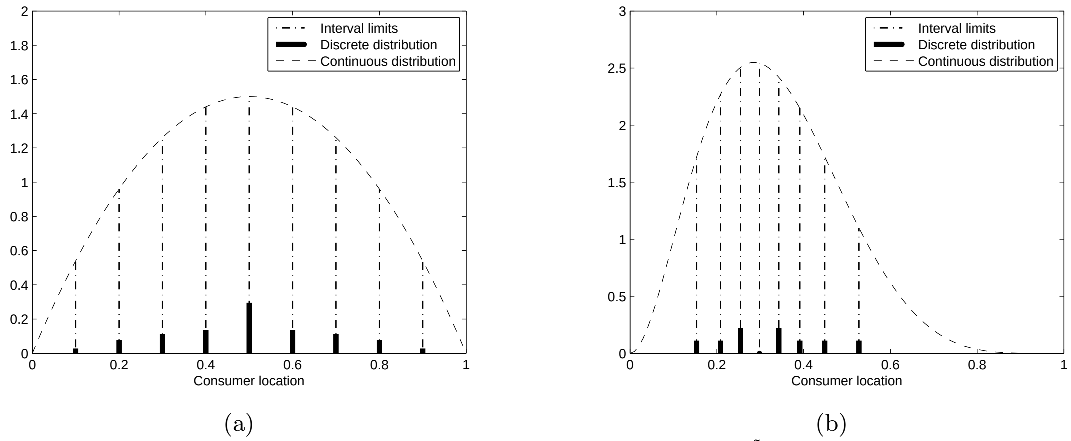


Figure 6 Examples of discretizing the consumer taste distribution: (a) $G^{\tilde{Z}}$, discretization of Beta(2,2) using 11 equidistant locations, defined in Proposition 3; (b) $G_5^{\tilde{Z}}$, a discretization of Beta(3,6) using 9 equiprobable intervals, defined in Proposition 4 (height of each solid line corresponds to g_i, the probabilities).

5. Numerical Study

The purpose of our numerical analysis is to (i) illustrate the rate of convergence of the lower and upper bounds, (ii) compare the relative performance of the four discretization methods used in computing these bounds, and (iii) determine how model parameters impact the performance of the discretization methods and identify when it is necessary to use a finer partition (i.e., a larger N).

First, we illustrate the convergence of the lower and upper bounds with an example. We use Beta(1,6) and Beta(3,6) as the consumer taste distribution, F, and set $\bar{p} = 25$, $c = 5$, $k = 4$, $m = 1$ and $d(z) = 40|z|$. Figure 7 plots the four lower bounds $\pi_D(N), \pi_P(N), \pi_S(N), \pi_A(N)$ and the four upper bounds $\tilde{\pi}_D(N), \tilde{\pi}_P(N), \tilde{\pi}_S(N), \tilde{\pi}_A(N)$ as a function of $N \in \{1, \ldots, 100\}$. (These two graphs are representative of convergence graphs we obtained using different distributions and parameters.) We see that the convergence of the upper bounds tends to be slower than that of the lower bounds, which suggests that the lower bounds may be closer to the optimal profit π^* than the optimality gaps calculated below indicate. This makes our performance analysis conservative.

Next, to study the performance of the discretization methods, we generate 252 problem instances. We set $\bar{p} = 25$, $c = 5$ and $m = 1$; vary the fixed cost $k \in \{1, 2, 3, 4\}$ and the transportation cost $d(z) = d|z|$ with $d \in \{20, 40, 80\}$; and consider Beta distributions with parameters (1,1), (1,1.1),

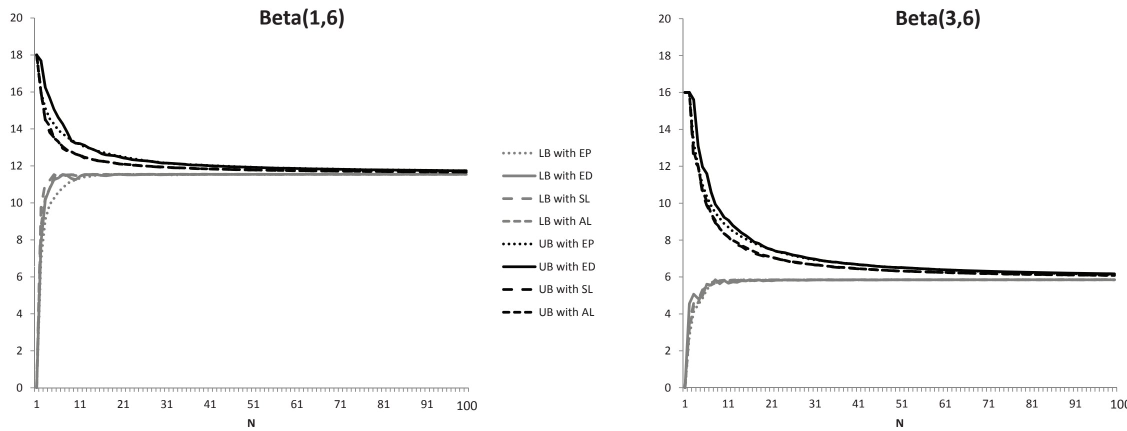


Figure 7 Convergence of the lower and upper bounds using the four discretization methods for two problem instances with Beta(1,6) and Beta(3,6) ($\bar{p}=25$, $c=5$, $k=4$, $m=1$ and $d(z)=40|z|$).

(1,1.4), (1,1.8), (1,2), (1,4), (1,6), (3,3), (6,6), (9,9), (12,12), (15, 15), (3,6), (3,9), (3,12), (6,9), (6,12), (9,12) as well as triangular distributions on $[0,1]$ with modes at 0, 0.25 and 0.5. All 21 of these distributions are unimodal; 7 of them are monotone, 7 are symmetric, and 7 are asymmetric and non-monotone.

We calculate optimality gaps as follows. For distributions that are symmetric or monotone, we use the ED and EP discretization methods with 10,000 intervals to obtain an upper bound as in Proposition 3 and calculate the upper bound estimate as $UB=\min\{\tilde{\pi}_D(10,000),\tilde{\pi}_P(10,000)\}$. Similarly, for asymmetric non-monotone distributions, we use Proposition 4 with 500 intervals to calculate the upper bound estimate $UB=\min\{\tilde{\pi}_D(500),\tilde{\pi}_P(500)\}$. We use fewer intervals in latter cases because of the increased complexity of the upper bound calculation – $O(N^3)$ versus $O(N^2)$ – as shown in §4.2. For all problem instances and for all N values listed in the tables, we calculate the percentage optimality gap of a particular lower bound on profit, LB, as $\frac{UB-LB}{UB}\times 100$. The optimality gap provides an estimate of the maximum percentage difference between the optimal profit π^* and the lower bound on profit.

Tables 1 and 2 compare the average optimality gap of the four discretization methods as a function of N. Using 10,000 (as opposed to 500) intervals in Table 1 to calculate the upper bound for each problem instance naturally results in tighter upper bounds and lower optimality gaps

	Optimality gaps (%)				Computation time (seconds)			
N	ED	EP	SL	AL	ED	EP	SL	AL
5	7.981%	12.204%	10.472%	11.206%	0.00	0.00	0.39	0.05
6	6.948%	10.103%	5.975%	6.540%	0.00	0.01	0.46	0.41
7	7.646%	8.177%	4.441%	6.055%	0.00	0.01	0.67	0.63
8	3.373%	6.141%	3.547%	4.565%	0.00	0.01	0.97	0.94
9	3.490%	5.665%	3.433%	3.421%	0.00	0.01	1.35	1.31
10	2.949%	4.067%	1.870%	2.572%	0.00	0.01	2.38	2.38
50	0.170%	0.232%	0.175%	0.144%	0.07	0.07	69.27	11.55
100	0.099%	0.100%	0.079%	0.085%	0.22	0.24	1562.27	359.87
500	0.059%	0.060%			5.42	5.55		
1000	0.059%	0.059%			21.90	21.99		
5000	0.058%	0.058%			631.13	778.18		
10000	0.058%	0.058%			2988.00	2990.33		

Table 1 Average optimality gaps and computation time of lower bounds as a function of N for 14 monotone or symmetric unimodal distributions ($\bar{p}=25,\ c=5,\ k\in\{1,2,3,4\},\ m=1$ and $d(z)=d|z|$ with $d\in\{20,40,80\}$).

	Optimality gaps (%)				Computation time (seconds)			
N	ED	EP	SL	AL	ED	EP	SL	AL
5	9.275%	15.684%	10.734%	12.397%	0.00	0.00	0.63	0.09
6	8.756%	13.232%	6.964%	8.658%	0.00	0.00	0.83	0.13
7	8.211%	10.732%	6.029%	7.304%	0.00	0.00	1.23	0.20
8	7.129%	8.993%	5.500%	5.896%	0.00	0.00	1.76	0.29
9	5.490%	7.803%	4.423%	5.333%	0.00	0.00	2.41	0.41
10	4.441%	6.822%	3.588%	4.399%	0.00	0.01	3.89	0.63
50	1.175%	1.353%	1.093%	1.105%	0.06	0.06	195.99	70.72
100	1.025%	1.067%	1.025%	1.012%	0.22	0.23	2134.35	643.38
500	0.991%	0.992%			5.24	5.25		
1000	0.990%	0.990%			21.12	21.20		
5000	0.989%	0.989%			620.75	621.01		
10000	0.989%	0.989%			2927.23	2924.85		

Table 2 Average optimality gaps and computation time of lower bounds as a function of N for 7 non-monotone, asymmetric unimodal distributions ($\bar{p}=25,\ c=5,\ k\in\{1,2,3,4\},\ m=1$ and $d(z)=d|z|$ with $d\in\{20,40,80\}$).

compared to Table 2. This does not indicate that the performance of the discretization methods is worse with asymmetric unimodal distributions. In fact, our experience is that the optimality gaps are similar if one uses comparable upper bounds. We focus more on Table 1 in our optimality gap discussion, because it is more representative of the performance of our discretization methods.

We observe that all four methods provide a very accurate solution for $N=50$ with an optimality gap of less than 0.3% (see Table 1). As expected, SL and AL methods perform well. These methods generate finer intervals around the mode of the consumer taste distribution allowing the best

		ED	EP	SL	AL
k	1	0.04%	0.08%	0.04%	0.04%
	2	0.07%	0.07%	0.06%	0.06%
	3	0.08%	0.09%	0.07%	0.07%
	4	0.22%	0.18%	0.15%	0.18%
d	20	0.02%	0.06%	0.02%	0.02%
	40	0.05%	0.06%	0.05%	0.05%
	80	0.25%	0.19%	0.18%	0.20%

Table 3 Average optimality gaps of lower bounds as a function of k and d for 7 monotone and 7 symmetric unimodal distributions ($\bar{p}=25$, $c=5$, $k\in\{1,2,3,4\}$, $m=1$, $d(z)=d|z|$ with $d\in\{20,40,80\}$, and $N=100$).

solution from these discretization methods to match up closely with the optimal solution, which places products with narrow market segments around the mode as we show in Theorem 1c (see Figure 8 for an illustrative example). It is surprising that the EP method performs consistently worse than all the other methods, especially the ED method, despite the fact that it uses the consumer taste distribution unlike the ED method. The example in Figure 8 and other examples we saw suggest that the EP method may generate much wider intervals on either end of the product space than the other three methods. Because these intervals still capture a significant portion of the probability mass, they might be covered in the EP solution but yield little profit, because wide intervals mean low prices. In comparison, the corresponding intervals in the other three methods are generally narrower, and they may not be covered so that the solution resembles the best assortment more closely. Eventually, for high N, all methods yield near-optimal solutions.

Tables 1 and 2 also show the average computation times as a function of N.[1] While generating the set $\hat{Z}$ for the EP and ED methods is virtually immediate, doing so for the SL and AL methods requires a time-consuming iterative procedure. We can obtain a near-optimal lower bound in under a minute using EP or ED with $N=1,000$ intervals, but we could do at most $N=100$ with the SL and AL methods (the remaining calculations would take more than 5 hours per problem).

In an effort to identify when to use a finer partition, Table 3 focuses on 168 of our problem instances with monotone and symmetric distributions and shows optimality gaps for $N=100$ as a function of the fixed cost k and the transportation cost d. We see that an increase in the fixed cost

[1] We used a X220 Tablet Lenovo Thinkpad with a 2.80 GHz processor and 8 GB of RAM.

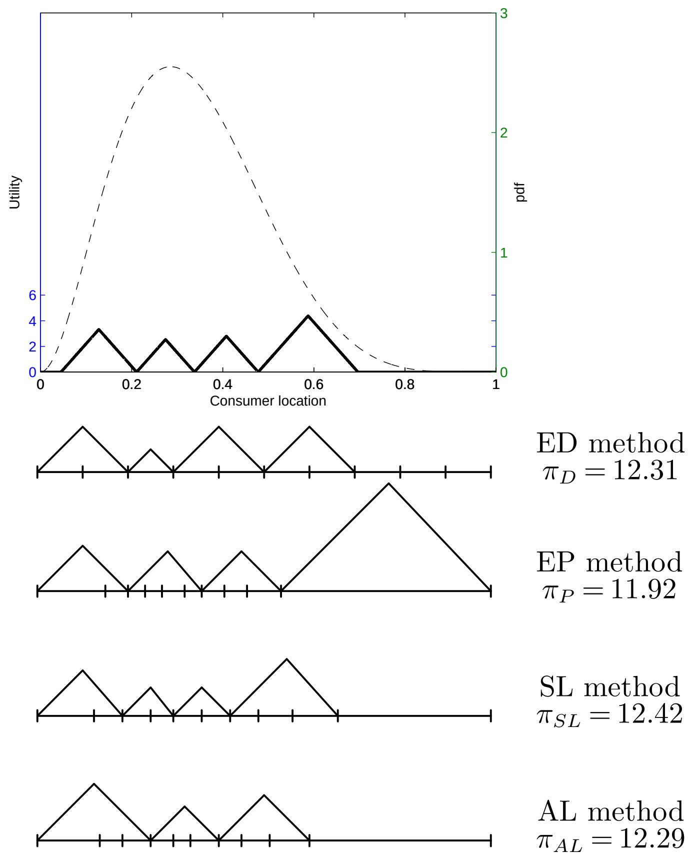


Figure 8 Comparison of best assortments via the four discretization methods with the optimal assortment ($\pi^* = 12.63$) where f is Beta(3,6), $\bar{p} = 25$, $c = 5$, $k = 1$, $m = 1$, $d(z) = 40|z|$ and $N = 10$.

or the transportation cost leads to larger optimality gaps for the ED, SL and AL methods. This is because higher values of k and d generally lead to a smaller total market share for the optimal assortment. Hence, in the optimal solution, the covered part of the product space is partitioned into fewer intervals as k and d increase, making the discretization methods less precise.

In conclusion, our results suggest that one does not need to use a very large value of N in order to get a good estimate of the optimal profit. This is especially true when k and d are low. Although these numerical results are restricted to unimodal distributions, our investigations using more general distributions suggest – judging from the speed of convergence of the lower bounds – that these conclusions apply more generally. The performance of discretization methods with as few as 10 intervals is very encouraging. In multi-period assortment planning problems, where the calculation of the optimal assortment has to be repeated at each stage of a DP and the total computational time increases rapidly with N as in Ulu et al. (2012), we advocate using the ED method which combines good performance with short computational time even though it does not use any information about the consumer taste distribution.

6. Conclusion

The Hotelling-Lancaster locational choice model has been used extensively in product line design problems to model consumer preferences for horizontally differentiated products. This model presents many advantages such as a convenient visual interpretation of the product space and tractable utility-based expressions for purchase probabilities and market shares.

In this paper we study optimal product line design to achieve horizontal product differentiation. We use a generalization of the basic Hotelling-Lancaster model to (1) any distribution for consumer tastes (typically, uniform distribution is assumed), (2) an asymmetric convex function for substitution disutility (typically, a symmetric linear or quadratic function is assumed), and (3) the possibility of not covering the market fully (typically, full coverage is assumed).

In this general setting, we find that the optimal assortment has a lot of structure especially if the consumer taste distribution is unimodal: a product that carves itself a higher market share also commands a higher price, and those high-ticket popular products are positioned close to the mode. We also prove a novel monotonicity result: as the consumer taste distribution shifts towards one end of the product space, the optimal assortment cannot move in the opposite direction.

To solve for the optimal assortment for any consumer taste distribution, we develop a number of lower bounds using discretizations of the product space and upper bounds using discretizations of the consumer taste distribution, all of which converge to the optimal profit as the discretizations become finer. We describe a shortest path method to calculate these bounds efficiently.

We find that the average optimality gap of the lower bounds is less than 0.1% when the product space is partitioned into more than 100 intervals, suggesting that the convergence to the optimal profit is quite fast. Moreover, using distributional information when discretizing does not payoff:

equidistant discretization works remarkably well in comparison to the other three methods we tested that do depend on the consumer taste distribution.

Because of its numerous advantages in applicability and practicality, the locational choice model is attractive for exploring new directions in assortment planning. For example, the properties of optimal assortments and the solution methodology we develop in this paper can be helpful in devising heuristic solutions for multi-period dynamic assortment planning problems, which are well known to pose computational challenges (Rusmevichientong et al. 2010, Ulu et al. 2012).

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Appendix A: Proofs

Proof of Lemma 1

We prove the result by a series of contradiction arguments. First, contiguity of market segments is implied by the following simple fact. If the consumers located at y_j and $y_{j'}$ ($y_j < y_{j'}$) optimally buy product i, so do all the consumers located at any point in $(y_j, y_{j'})$. So, any form of non-contiguity would create a contradiction; market segments are closed intervals in $\mathbb{R}$.

Second, suppose the optimal assortment includes a product i such that $\underline{x}_i = \overline{x}_i$. Because $F(\cdot)$ is a continuous distribution, the market share of this product would be zero, hence the firm would prefer not to offer product i and save the fixed cost, which contradicts optimality.

Third, suppose the optimal assortment includes a product i such that $x_i < L(\underline{x}_i, \overline{x}_i)$. We have $U(\underline{x}_i, x_i, p_i) > U(\overline{x}_i, x_i, p_i) \geq 0$ by definition of market segments. Since the consumers located at $\underline{x}_i$ get a strictly positive utility from product i, it is possible to increase p_i while keeping a_i (and the rest of the market segments) exactly the same by appropriately shifting x_i to the right. Hence, the current solution could not be optimal. (The case with $x_i > L(\underline{x}_i, \overline{x}_i)$ can be treated similarly.)

Finally, suppose the optimal solution contains a product i with price $p_i < \bar{p} - T(\underline{x}_i, \overline{x}_i)$. (The case with $>$ would imply that the consumers at the boundaries of the market segment get strictly negative utility from product i, which is not possible by definition of the market segments.) This implies that $U(\underline{x}_i, x_i, p_i) = U(\overline{x}_i, x_i, p_i) > 0$. Since consumers located at $\underline{x}_i$ and $\overline{x}_i$ get a strictly positive utility from product i, it is possible to increase p_i while keeping x_i and a_i (and the rest of the market segments) exactly the same. Hence, we have a contradiction. Q.E.D.

Proof of Lemma 2

Given a positive constant $\xi > 0$, define $L(\xi) \in [0, \xi]$ as the unique solution to the equation $d(L(\xi)) = d(L(\xi) - \xi)$, and let $T(\xi) = d(L(\xi)) = d(L(\xi) - \xi)$.

Differentiating both sides of $d(L(\xi)) = d(L(\xi) - \xi)$, we obtain

$$L'(\xi) = \frac{d'(L(\xi) - \xi))}{d'(L(\xi) - \xi)) - d'(L(\xi))}$$

which satisfies $0 < L'(\xi) < 1$, because $d'(L(\xi) - \xi)) < 0$ and $d'(L(\xi)) > 0$. Using the definition of T, we also have $T'(\xi) = d'(L(\xi)) \cdot L'(\xi)$ and

$$T''(\xi) = \frac{d''(L(\xi)) \cdot L'(\xi) \cdot [d'(L(\xi) - \xi)]^2 + d''(L(\xi) - \xi) \cdot [1 - L'(\xi)] \cdot [d'(L(\xi))]^2}{[d'(L(\xi) - \xi) - d'(L(\xi))]^2}$$

both of which are positive, because $d'(L(\xi)) > 0$, $0 < L'(\xi) < 1$, and $d(\cdot)$ is convex. Q.E.D.

Proof of Lemma 3

Suppose $e \le (a+b)/2$. (The proof for $e \ge (a+b)/2$ holds by symmetry.) The contribution to optimal profit by the two products that cover $[a,x]$ and $[x,b]$ as a function of x, and its derivative with respect to x, can be written as follows:

$$\pi_{[a,b]}(x) = [\bar{p} - T(x-a) - c][F(x) - F(a)] + [\bar{p} - T(b-x) - c][F(b) - F(x)]$$
$$\pi'_{[a,b]}(x) = -T'(x-a)[F(x) - F(a)] + T'(b-x)[F(b) - F(x)] + [T(b-x) - T(x-a)]f(x)$$

We are interested in finding out if the profit can be improved by shifting x, while fixing a and b, and adjusting the product locations and prices accordingly. Note that such a shift would have no impact on the rest of the product line.

Case 1. Suppose $x \in [a,e]$. Because $[a,x]$ is narrower than $[x,b]$, Lemma 2 implies that $T(x-a) < T(b-x)$ and $0 < T'(x-a) < T'(b-x)$. Replacing $T'(b-x)$ with $T'(x-a)$ in $\pi'_{[a,b]}(x)$, we obtain

$$\pi'_{[a,b]}(x) > T'(x-a)[F(a) + F(b) - 2F(x)] + [T(b-x) - T(x-a)]f(x).$$

The right-hand-side is positive, because $T'(x-a) > 0$ by Lemma 2, $F(a) + F(b) - 2F(x) > 0$ by the hypothesis that $x \in [a,e]$, and $T(b-x) - T(x-a) > 0$. Therefore, $\pi'_{[a,b]}(x) > 0$ for all $x \in [a,e]$, and any $x \in [a,e]$ cannot occur in the optimal solution.

Case 2. Suppose $x \in [(a+b)/2, b]$. Because $[a,x]$ is a wider segment than $[x,b]$, Lemma 2 implies that $T(x-a) > T(b-x)$ and $T'(x-a) > T'(b-x) > 0$. Replacing $T'(b-x)$ with $T'(x-a)$ in $\pi'_{[a,b]}(x)$, we obtain

$$\pi'_{[a,b]}(x) < T'(x-a)[F(a) + F(b) - 2F(x)] + [T(b-x) - T(x-a)]f(x).$$

The right-hand-side is negative, because $T'(x-a) > 0$ by Lemma 2, $F(a) + F(b) - 2F(x) < 0$ by the hypothesis that $x \in [(a+b)/2, b]$, and $T(b-x) - T(x-a) < 0$. Therefore, $\pi'_{[a,b]}(x) < 0$ for all $x \in [(a+b)/2, b]$, and any $x \in [(a+b)/2, b]$ cannot occur in the optimal solution. It follows from cases 1 and 2 that $x \in [e, (a+b)/2]$ must be true for the current assortment to be optimal. Q.E.D.

Proof of Theorem 1

Part a. The proof is by contradiction. Suppose the mode M is not covered in $\mathbf{a}^*$. Find product i to the left of M that minimizes $M - \overline{x}_i^*$. It must be that the segment $(\overline{x}_i^*, M]$ is not covered. Shifting the location of product i to the right – while keeping its market segment length and price the same – such that its upper boundary point coincides with the mode, $\overline{x}_i^* = M$, would strictly improve the profit. This follows from unimodality of F; the market share of product i would increase, hence its profit contribution. Note that this shift has no impact for the rest of the assortment.

Part b. The proof is by contradiction. Suppose $\mathbf{a}^*$ contains two consecutive products i and $i+1$ to the left of M such that $\overline{x}_i^* < \underline{x}_{i+1}^*$ and $\underline{x}_{i+1}^* < M$. Therefore, consumers in $(\overline{x}_i^*, \underline{x}_{i+1}^*)$ do not purchase. Shifting product i to the right – while keeping its market segment length and price the same – such that its market segment becomes adjacent to that of product $i+1$, i.e., $\overline{x}_i^* = \underline{x}_{i+1}^*$, would strictly improve the profit. This follows from unimodality of F; the market share of product i would increase, hence its profit contribution. Note that this shift has no impact for the rest of the assortment. (The proof works similarly for products on the right-side of the mode.)

Part c. Suppose the optimal assortment has two products that cover adjacent market segments $[a,b]$ and $[b,c]$ to the right of the mode ($a \geq M$). Let e be the point that divides $[a,c]$ into two equiprobable intervals, i.e., $F(e) - F(a) = F(c) - F(e)$. Because F is unimodal and $[a,c]$ lies to the right of the mode, we have $e < (a+c)/2$. Then, Lemma 3 implies that $b \in [e, (a+c)/2]$. This in turn implies that $[b,c]$, the segment further away from the mode, is wider *and* has a smaller market share than $[a,b]$. By Lemma 2, a wider segment means a lower price. The proof for products with market segments that lie to the left of the mode in their entirety holds by symmetry.

If f is monotone decreasing (increasing), then M is the lower (upper) boundary of the product space, hence the argument above is sufficient to establish the result. To prove that prices and market shares are unimodal for any unimodal f, we need to rule out the following possibility: the product that covers the mode has a lower price and a lower market share than both products adjacent to it. Suppose the optimal assortment has three market segments $[a,b]$, $[b,c]$ and $[c,d]$, where $M \in [b,c]$, that satisfy the above property. Lemma 3 implies that $b \in [e_{[a,c]}, (a+c)/2]$ and $c \in [(b+d)/2, e_{[b,d]}]$, where $e_{[a,c]}$ and $e_{[b,d]}$ are the points that divide segments $[a,c]$ and $[b,d]$ each into equiprobable intervals. By the mean value theorem, there must exist $x_0 \in [b,c]$ such that $F(c) - F(b) = f(x_0)(c-b)$. For segments $[a,b]$ and $[b,c]$ we then have the following relationships:

$$F(c) - F(b) = f(x_0)(c-b) > f(x_0)(b-a)$$
$$F(c) - F(b) < F(b) - F(a) < (b-a)f(b)$$

which follow from $M \in [b,c]$, $[b,c]$ is wider and has less market share than $[a,b]$, and F is increasing in $[a,b]$. These imply that $f(x_0) < f(b)$. Similarly, for segments $[b,c]$ and $[c,d]$ we have

$$F(c) - F(b) = f(x_0)(c-b) > f(x_0)(d-c)$$
$$F(c) - F(b) < F(d) - F(c) < (d-c)f(c)$$

which imply that $f(x_0) < f(c)$. This is a contradiction; we cannot have both $f(x_0) < f(b)$ and $f(x_0) < f(c)$, because $x_0 \le M$ implies $f(x_0) > f(b)$ and $x_0 > M$ implies $f(x_0) > f(c)$. Therefore, $[b,c]$ cannot have larger length and smaller market share than both $[a,b]$ and $[c,d]$.

Finally, to establish the result in the symmetric unimodal case, we take the domain and mode of f to be $[0,1]$ and $M = 0.5$. Suppose that the product with the highest price is to the left of the product that covers M (the case where it is to the right is identical). Let $[a,b]$ and $[b,c]$ be the market segments of these two products, and $M \in [b,c]$. We have $F(c) - F(b) > F(2b-a) - F(b)$, because $b-a < c-b$ implies $c > 2b-a$. If $2b-a \le M$, then we directly have $F(2b-a) - F(b) \ge F(b) - F(a)$ since segments $[a,b]$ and $[b, 2b-a]$ have the same length but segment $[b, 2b-a]$ is closer to the mode. Otherwise, if $2b-a > M$, we write:

$$\begin{aligned}
F(2b-a) - F(b) &= [F(2b-a) - F(M)] + [F(M) - F(b)] \\
&= [F(M) - F(1-2b+a)] + [F(M) - F(b)] \\
&\ge [F(2b-M) - F(a)] + [F(M) - F(b)] \\
&\ge [F(2b-M) - F(a)] + [F(b) - F(2b-M)] = F(b) - F(a)
\end{aligned}$$

The second equality follows from the symmetry of F: in general $F(y)-F(x)=F(1-x)-F(1-y)$ and $M=1-M$. The inequalities come from the following observations: segments $[a,2b-M]$ and $[1-2b+a,M]$ have the same length, but segment $[1-2b+a,M]$ is closer to the mode; segments $[2b-M,b]$ and $[b,M]$ have the same length, but segment $[b,M]$ is closer to the mode. Thus, we conclude that $F(b)-F(a)<F(c)-F(b)$, but this creates a contradiction with Lemma 3, which implies that $F(b)-F(a)\geq F(c)-F(b)$ because we must have $b\in[e_{[a,c]},(a+c)/2]$. Q.E.D.

Proof of Theorem 2

We first need the following two intermediate results:

LEMMA 5 **(Lemma 1 in Strulovici and Quah, 2009.)**. *Suppose $[x^L,x^H]$ is a compact interval of $\mathbb{R}$, and that α and h are real valued functions defined on $[x^L,x^H]$, with h integrable and α increasing (and thus integrable as well). If $\int_{\hat{x}}^{x^H} h(x)\,dx\geq 0$ for all $\hat{x}$ in $[x^L,x^H]$, then*

$$\int_{x^L}^{x^H}\alpha(x)h(x)\,dx\geq\alpha(x^L)\int_{x^L}^{x^H}h(x)\,dx$$

LEMMA 6. *Suppose that the optimal assortment under F is $\mathbf{a}^*$ with n products, and that $\mathbf{a}^*\succeq_R\mathbf{a}$. Then, $\mathbf{a}^*$ has a higher profit contribution than $\mathbf{a}$ from all consumers in $[\hat{x},x^H]$, i.e.,*

$$\int_{\hat{x}}^{x^H}\rho(x,\mathbf{a}^*)f(x)\,dx\geq\int_{\hat{x}}^{x^H}\rho(x,\mathbf{a})f(x)\,dx,\ \text{for all } x_L\leq\hat{x}\leq x^H,$$

where $\rho(x,\mathbf{a})=p_i(a_i)-c$ if $x\in a_i$ for some $i\leq n$, 0 otherwise, represents the profit margin on sales (if any) to consumers in location x.

Proof. Notice that if $\hat{x}=x_L$, this property is equivalent to $\pi(\mathbf{a}^*,F)\geq\pi(\mathbf{a},F)$, which holds for $\mathbf{a}^*$ is the optimal assortment given F. We prove the property for any $\hat{x}$ by contradiction. Suppose

$$\int_{\hat{x}}^{x^H}\left(\rho(x,\mathbf{a}^*)-\rho(x,\mathbf{a})\right)f(x)\,dx<0. \tag{5}$$

If this is true, we can construct an assortment $\mathbf{a}_{\hat{x}}$ such that it has (weakly) higher profit contributions than $\mathbf{a}=([\underline{x}_1,\overline{x}_1],\ldots,[\underline{x}_n,\overline{x}_n])$ for any location $x\geq\hat{x}$ and (weakly) higher profit contributions than $\mathbf{a}^*=([\underline{x}_1^*,\overline{x}_1^*],\ldots,[\underline{x}_n^*,\overline{x}_n^*])$ for any location $x<\hat{x}$. There are four cases to consider:

1. $\hat{x}$ is not covered under $\mathbf{a}$ or $\mathbf{a}^*$. Then, there exist two products i and i^* in $\mathbf{a}$ and $\mathbf{a}^*$, respectively, such that $i^* \leq i$, $\overline{x}_i < \hat{x} < \underline{x}_{i+1}$, and $\overline{x}^*_{i^*} < \hat{x} < \underline{x}^*_{i^*+1}$. Let $\mathbf{a}_{\hat{x}}$=($[\underline{x}^*_1, \overline{x}^*_1]$,…,$[\underline{x}^*_{i^*}, \overline{x}^*_{i^*}]$, $[\underline{x}_{i+1}, \overline{x}_{i+1}]$,…,$[\underline{x}_n, \overline{x}_n]$).

2. $\hat{x}$ is not covered under $\mathbf{a}$ but it is covered under $\mathbf{a}^*$. Then, there exist two products i and i^* in $\mathbf{a}$ and $\mathbf{a}^*$, respectively, such that $i^* \leq i$, $\overline{x}_i < \hat{x} < \underline{x}_{i+1}$, and $\underline{x}^*_{i^*} \leq \hat{x} \leq \overline{x}^*_{i^*}$. Let $\mathbf{a}_{\hat{x}} = ([\underline{x}^*_1, \overline{x}^*_1], \ldots, [\underline{x}^*_{i^*}, \hat{x}], [\underline{x}_{i+1}, \overline{x}_{i+1}], \ldots, [\underline{x}_n, \overline{x}_n])$.

3. $\hat{x}$ is covered under $\mathbf{a}$ but it is not covered under $\mathbf{a}^*$. Then, there exist two products i and i^* in $\mathbf{a}$ and $\mathbf{a}^*$, respectively, such that $i^* \leq i$, $\underline{x}_i \leq \hat{x} \leq \overline{x}_i$, and $\overline{x}^*_{i^*-1} < \hat{x} < \underline{x}^*_{i^*}$. Let $\mathbf{a}_{\hat{x}} = ([\underline{x}^*_1, \overline{x}^*_1], \ldots, [\underline{x}^*_{i^*-1}, \overline{x}^*_{i^*-1}], [\hat{x}, \overline{x}_i], \ldots, [\underline{x}_n, \overline{x}_n])$.

4. $\hat{x}$ is covered under both $\mathbf{a}$ and $\mathbf{a}^*$. Then, there exist two products i and i^* in $\mathbf{a}$ and $\mathbf{a}^*$, respectively, such that $i^* \leq i$, $\underline{x}_i \leq \hat{x} \leq \overline{x}_i$, and $\underline{x}^*_{i^*} \leq \hat{x} \leq \overline{x}^*_{i^*}$. If $i^* = i$, let $\mathbf{a}_{\hat{x}}$=($[\underline{x}^*_1, \overline{x}^*_1]$,…,$[\underline{x}^*_{i^*-1}, \overline{x}^*_{i^*-1}]$, $[\underline{x}^*_{i^*}, \overline{x}_i]$,…,$[\underline{x}_n, \overline{x}_n]$). If $i^* < i$, let $\mathbf{a}_{\hat{x}} = ([\underline{x}^*_1, \overline{x}^*_1], \ldots, [\underline{x}^*_{i-1}, \overline{x}^*_{i-1}], [\underline{x}^*_i, \hat{x}], [\hat{x}, \overline{x}_i], \ldots, [\underline{x}_n, \overline{x}_n])$.

In all the cases considered above, it is easy to see that $\mathbf{a}_{\hat{x}}$ has n products or less and satisfies $\rho(x, \mathbf{a}_{\hat{x}}) - \rho(x, \mathbf{a}) \geq 0 \quad \forall x \geq \hat{x}$, and $\rho(x, \mathbf{a}_{\hat{x}}) - \rho(x, \mathbf{a}^*) \geq 0 \quad \forall x < \hat{x}$, which imply that

$$\int_{\hat{x}}^{x^H} (\rho(x, \mathbf{a}_{\hat{x}}) - \rho(x, \mathbf{a}))\, f(x)\, dx \geq 0 \tag{6}$$

$$\int_{x_L}^{\hat{x}} (\rho(x, \mathbf{a}_{\hat{x}}) - \rho(x, \mathbf{a}^*))\, f(x)\, dx \geq 0 \tag{7}$$

(5) and (6) imply that $\int_{\hat{x}}^{x^H} (\rho(x, \mathbf{a}_{\hat{x}}) - \rho(x, \mathbf{a}^*))\, f(x)\, dx > 0$. This inequality together with (7) imply that $\int (\rho(x, \mathbf{a}_{\hat{x}}) - \rho(x, \mathbf{a}^*))\, f(x)\, dx > 0$, which in turn implies $\pi(\mathbf{a}_{\hat{x}}, F) > \pi(\mathbf{a}^*, F)$ because $\mathbf{a}_{\hat{x}}$ has n or fewer products (i.e., fixed cost of $\mathbf{a}_{\hat{x}}$ is less than or equal to fixed cost of $\mathbf{a}^*$). This creates a contradiction because $\mathbf{a}^*$ is the optimal assortment for F. Q.E.D.

We now proceed with the proof of the theorem. We need to show that if $\mathbf{a}^*$ is optimal for F, then $\pi(\mathbf{a}^*, G) \geq \pi(\mathbf{a}, G)$ for any assortment $\mathbf{a}$ and any consumer taste distribution G such that $\mathbf{a}^* \succeq_R \mathbf{a}$ and $G \succeq_{LR} F$. Note that,

$$\pi(\mathbf{a}^*, G) - \pi(\mathbf{a}, G) = \int [\rho(x, \mathbf{a}^*) - \rho(x, \mathbf{a})] g(x)\, dx = \int [\rho(x, \mathbf{a}^*) - \rho(x, \mathbf{a})] \frac{g(x)}{f(x)} f(x)\, dx.$$

Because $\frac{g(x)}{f(x)}$ is increasing and $\int_{\hat{x}}^{x^H}[\rho(x,\mathbf{a}^*)-\rho(x,\mathbf{a})]f(x)\,dx \geq 0$ (from Lemma 6), Lemma 5 implies

$$\int[\rho(x,\mathbf{a}^*)-\rho(x,\mathbf{a})]\frac{g(x)}{f(x)}f(x)\,dx \geq \frac{g(x^L)}{f(x^L)}\int[\rho(x,\mathbf{a}^*)-\rho(x,\mathbf{a})]f(x)\,dx$$

Because $\frac{g(x^L)}{f(x^L)} \geq 0$ and $\mathbf{a}^*$ is optimal for F, this implies $\pi(\mathbf{a}^*,G)-\pi(\mathbf{a},G)\geq 0$.   Q.E.D.

For use in the next four proofs, let $\hat{z}_i^N$ denote the i-th largest element of $\hat{Z}$ when the *product space* is partitioned in N intervals and $\tilde{z}_i^N$ denote the i-th largest element of $\tilde{Z}$ when the *consumer taste distribution* is discretized into $N+1$ locations.

Proof of Proposition 1

Given a value of N, we construct the "closest matching" assortment to the optimal assortment $\mathbf{a}^*$ as follows. For $j=1,...,n^*$, let $\underline{x}_j^N$ be the value of $\hat{z}_i$ that is closest to $\underline{x}_j^*$. And let $\overline{x}_j^N$ be the value of $\hat{z}_i^N$ that is closest to $\overline{x}_j^*$ and satisfies $\overline{x}_j^N > \underline{x}_j^N$. For small values of N, the number of products in the closest matching assortment may be less than n^* (because the market segments of multiple products from the optimal solution may be inside the same interval), but for N large enough, the number of products in the closest matching assortment is equal to n^*. In what follows we assume that N is large enough. As N gets larger, the interval lengths tend to zero under both ED and EP methods, i.e., $\hat{z}_i^N-\hat{z}_{i-1}^N \to 0$ as $N\to\infty$. It follows that the distance between a fixed point y and the closest $\hat{z}_i^N$ value decreases as $N\to\infty$. As a result, we have $\underline{x}_j^N \to \underline{x}_j^*$ and $\overline{x}_j^N\to\overline{x}_j^*$ as $N\to\infty$. The closest matching assortment is feasible for any given N ($\overline{x}_{j-1}^N \leq \underline{x}_j^N$ for $j=2,...,n^*$), so the profit it generates, $\hat{\pi}(N)$, is a lower bound on $\pi_D(N)$. We show that $\hat{\pi}(N)$ tends to π^* as $N\to\infty$, as a result $\pi_D(N)$ tends to π^*. For a given N, $\hat{\pi}(N)$ is equal to $\sum_{j=1}^{n^*}\left[\left(\bar{p}-T(\overline{x}_j^N-\underline{x}_j^N)-c\right)\left[F(\overline{x}_j^N)-F(\underline{x}_j^N)\right]m-k\right]$, which tends to the optimal profit $\pi^*=\sum_{j=1}^{n^*}\left[\left(\bar{p}-T(\overline{x}_j^*-\underline{x}_j^*)-c\right)\left[F(\overline{x}_j^*)-F(\underline{x}_j^*)\right]m-k\right]$ as $N\to\infty$, because $\underline{x}_j^N\to\underline{x}_j^*$ and $\overline{x}_j^N\to\overline{x}_j^*$.   Q.E.D.

Proof of Proposition 2

We prove that $\hat{z}_i^N \to 0$ as $N\to\infty$. The rest of the proof is identical to that of Proposition 1.

First, we prove that $\hat{z}_i^N$ is decreasing in N for all $i=1,...,N$. Let $\hat{z}_{i,j}^N$ denote the value of $\hat{z}_i^N$ in the

j-th iteration of the iterative calculations using equations (3) and (4) with N intervals. Similarly, let $\hat{y}_{i,j}^N$ denote the value of $\hat{y}_i$ in the i-th iteration. For the SL method, (3) and (4) become:

$$\hat{y}_{i,j}^N = \frac{\int_{\hat{z}_{i-1,j}^N}^{\hat{z}_{i,j}^N} xf(x)dx}{F(\hat{z}_{i,j}^N) - F(\hat{z}_{i-1,j}^N)} \text{ for } i = 1, ..., N \tag{8}$$

$$\hat{z}_{i,j}^N = \frac{\hat{y}_{i,j-1}^N + \hat{y}_{i+1,j-1}^N}{2} \text{ for } i = 1, ..., N-1 \tag{9}$$

For the AL method, the first equation is replaced with:

$$\hat{y}_{i,j}^N = F^{-1}\left(\frac{F(\hat{z}_{i-1,j}^N) + F(\hat{z}_{i,j}^N)}{2}\right) \text{ for } i = 1, ..., N \tag{10}$$

We show that $\hat{z}_{i,j}^N$ is decreasing in N for all j by induction. For $j = 1$ we have $\hat{z}_{i,1}^N = \frac{i}{N} \geq z_{i,1}^{N+1} = \frac{i}{N+1}$ because the initialization is done using the equidistant method. Now assume that $\hat{z}_{i,j}^N \geq \hat{z}_{i,j}^{N+1}$ for all $i = 1, ..., N$. From (10) and (8), we see that $\hat{y}_{i,j}^N$ is increasing in $\hat{z}_{i,j}^N$ and $\hat{z}_{i-1}^N$, and therefore $\hat{y}_{i,j}^N \geq \hat{y}_{i,j}^{N+1}$ for all $i = 1, ..., N$. From (9), we see that $\hat{z}_{i,j}^N$ is increasing in $\hat{y}_{i,j-1}^N$ and $\hat{y}_{i,j-1}^N$ and therefore $\hat{z}_{i,j+1}^N \geq \hat{z}_{i,j+1}^{N+1}$, which proves the induction.

For a given i, the sequence of $\hat{z}_i^N$ values is decreasing in N and we have $\hat{z}_i^N \geq 0$ for all N. Therefore, by the monotone convergence theorem, it converges to a limit as $N \to \infty$. Next we prove that this limit is zero. First we show that $\hat{z}_1^N \to 0$. Suppose (contradiction) that this is not the case, i.e., that $\hat{z}_1^N \to \delta_1 > 0$. Let $\tilde{N}$ be such that $|z_1^N - \delta_1| < \epsilon$ for all $N \geq \tilde{N}$. Consider $N > \tilde{N}$, we have $|z_1^N - \delta_1| < \epsilon$ and $|z_1^{N+1} - \delta_1| < \epsilon$. We write $\hat{z}_1^N \simeq \hat{z}_1^{N+1} \simeq \delta_1$ where $\simeq$ means "identical up to a constant which goes to zero as N goes to ∞". From (10) or (8), this implies that $y_1^N \simeq y_1^{N+1}$. From (9), we get that $y_2^N \simeq y_2^{N+1} \simeq 2\delta_1 - y_1^N$. Because F is continuous, this must mean that $\hat{z}_2^N \simeq \hat{z}_2^{N+1}$. Repeating the argument until N, we get $\hat{y}_N^N \simeq \hat{y}_N^{N+1} < 1$ and $1 = \hat{z}_N^N \simeq \hat{z}_N^{N+1}$. However this would imply that $\hat{z}_{N+1}^{N+1} \simeq \hat{z}_N^{N+1} \simeq \hat{y}_{N+1}^{N+1}$ which would violate $\hat{z}_N^{N+1} \simeq \frac{\hat{y}_{N+1}^N + \hat{y}_{N+1}^{N+1}}{2}$. Hence we have a contradiction.

Now suppose that $\hat{z}_2^N \to \delta_2 > 0$. Since $z_1^N \to 0$, the first interval vanishes and the analysis above eventually applies to $\hat{z}_2^N$. Thus, it must be that $\hat{z}_2^N \to 0$, and in general $\hat{z}_i^N \to 0$ for all i. Q.E.D.

Proof of Propositions 3 and 4

It suffices to identify an assortment $\mathbf{a}$ that is feasible for the discrete problem under $G^{\tilde{Z}}$ (for Proposition 3) or any one of the $N+1$ distributions $G_j^{\tilde{Z}}$ (for Proposition 4) and yet more profitable than the optimal assortment $\mathbf{a}^*$ for the continuous distribution F. Let i_0 be the product that has the highest price and market share in $\mathbf{a}^*$. Let intervals l and u contain the boundary points of its market segment, i.e., $\underline{x}_{i_0}^* \in [\tilde{z}_{l-1}, \tilde{z}_l]$ and $\overline{x}_{i_0}^* \in [\tilde{z}_{u-1}, \tilde{z}_u]$. Recall that interval m contains the mode, i.e., $M \in [\tilde{z}_{m-1}, \tilde{z}_m)$. We first specify which discrete distribution to use to construct $\mathbf{a}$ depending on f and the relative positions of intervals l, u, and m.

If f is symmetric unimodal or monotone, then use $G^{\tilde{Z}}$. Otherwise, if f is asymmetric unimodal, then consider the following three distinct cases.

Case 1: Product i_0 covers the mode M. There are four subcases to consider: (a) interval m does not contain $\underline{x}_{i_0}^*$ or $\overline{x}_{i_0}^*$ ($l<m<u$); (b) interval m only contains $\underline{x}_{i_0}^*$ ($l=m<u$); (c) interval m only contains $\overline{x}_{i_0}^*$ ($l<m=u$); and (d) interval m contains both $\underline{x}_{i_0}^*$ and $\overline{x}_{i_0}^*$ ($l=m=u$). Use $G_m^{\tilde{Z}}$ in case 1b, and $G_0^{\tilde{Z}}$ in cases 1a, 1c, and 1d.

Case 2: Product i_0 covers a segment to the left of M, i.e., $\overline{x}_{i_0}^* < M$. There are three subcases to consider: (a) interval u does not contain M and $l<u<m$; (b) interval u does not contain M and $l=u<m$; (c) interval u contains M ($l \leq u = m$). Use $G_u^{\tilde{Z}}$ in case 2a, and $G_0^{\tilde{Z}}$ in cases 2b and 2c.

Case 3: Product i_0 covers a segment to the right of M, i.e., $\underline{x}_{i_0}^* > M$. There are three subcases to consider: (a) interval l does not contain M and $m<l<u$; (b) interval l does not contain M and $m<l=u$; (c) interval l contains M ($m=l\leq u$). Use $G_l^{\tilde{Z}}$ for 3a, $G_0^{\tilde{Z}}$ for 3b, and $G_m^{\tilde{Z}}$ for 2c.

We use the appropriate discrete distribution to determine the $\tilde{y}$ points (as defined in the propositions). We then construct assortment $\mathbf{a}$ as follows. Offer no products to cover those intervals that $\mathbf{a}^*$ does not cover at all. Offer one dedicated product (priced at $\bar{p}$) to cover the $\tilde{y}$ point for those intervals that contain one or more market segments of $\mathbf{a}^*$ in their entirety. For all the remaining intervals, identify groups of $\tilde{y}$ points that fall inside each of the market segments covered by $\mathbf{a}^*$, and offer one product for each group that covers exactly those points.

In all of the above cases, assortment $\mathbf{a}$ will have the same number of products as $\mathbf{a}^*$ or fewer, and every consumer who rationally buys one of its products will be charged a higher price than they would be under $\mathbf{a}^*$ (in weak sense). Therefore, assortment $\mathbf{a}$ is more profitable than $\mathbf{a}^*$, which implies that the optimal solution to the assortment problem under the discrete distribution constitutes an upper bound to the optimal profit π^* for the continuous distribution F.

The key is to make sure that everyone pays a higher price (in weak sense) under $\mathbf{a}$ than under $\mathbf{a}^*$. To see how this happens, suppose that an arbitrary interval $[\tilde{z}_{i-1}^N, \tilde{z}_i^N]$ lies to the right of the mode, i.e., $\tilde{z}_{i-1}^N > M$ (the case of an interval that lies to the left of the mode is similar). If there are multiple market segments in $\mathbf{a}^*$ that overlap with this interval, we know from Theorem 1c that the market segment on the far left (the one closest to M) pays the highest price. Therefore, moving all the probability mass to the lower boundary point $\tilde{z}_{i-1}^N$ of the interval can only increase the price paid in $\mathbf{a}$ by the consumers from interval $[\tilde{z}_{i-1}^N, \tilde{z}_i^N]$ compared to the price they pay in $\mathbf{a}^*$. Q.E.D.

Proof of Proposition 5

We showed in the proof of Propositions 1 and 2 that with the four discretization methods we have $\tilde{z}_i^N - \tilde{z}_{i-1}^N \to 0$ for all i as $N \to \infty$. Consider the assortment $\mathbf{a}$ constructed as explained in the proofs of Propositions 3 and 4. The fact that the interval lengths shrink as N becomes larger implies that, for every product j in the optimal assortment $\mathbf{a}^*$, $\tilde{z}_{i-1}^N \to \underline{x}_j^*$ or $\tilde{z}_i^N \to \underline{x}_j^*$ for some interval i, and $\tilde{z}_{i'-1}^N \to \overline{x}_j^*$ or $\tilde{z}_{i'}^N \to \overline{x}_j^*$ for some interval i'. Therefore, the market segments in assortment $\mathbf{a}$ tend to the market segments in $\mathbf{a}^*$, and the upper bound profit tends to the optimal profit π^*. Q.E.D.

Appendix B: Discrete Taste Distributions

Suppose that F is a discrete distribution over Ω with points of positive mass $Y = \{y_1, ...y_N\}$. Let θ_j be the probability that a randomly chosen consumer is at location y_j. Lemma 1 continues to hold in this case with the following modification to part (a): $\underline{x}_i^* \le \overline{x}_i^*$ for $i = 1, \dots, n$ and $\overline{x}_i^* < \underline{x}_{i+1}^*$ for $i = 1, \dots, n-1$. Further, the following lemma provides extra structure on the optimal assortment.

LEMMA 7. *Under a discrete distribution with points of positive mass $Y = \{y_1, ...y_N\}$, the boundary points of each market segment in the optimal assortment, $\underline{x}_i, \overline{x}_i$, belong to the set Y.*

Proof. We prove the result via a contradiction argument. Suppose the optimal solution includes a product i such that $\underline{x}_i \notin Y$ and $\overline{x}_i \in Y$ (the cases with $\underline{x}_i \in Y$ and $\overline{x}_i \notin Y$, and $\underline{x}_i, \overline{x}_i \notin Y$ can be treated similarly). Let y_j be the leftmost consumer location in $[\underline{x}_i, \overline{x}_i]$. (If none existed, the firm would prefer not to offer product i and save the fixed cost, which contradicts optimality.) We have $x_i \in [\underline{x}_i, \overline{x}_i]$ and $U(y_j, x_i, p_i) > 0$, so it is possible to increase p_i and shift x_i to the right just so that $\underline{x}_i$ becomes equal to y_j and $\overline{x}_i$ remains the same. Because product i attracts exactly the same market share as before and is offered at a higher price, the current solution could not have been optimal. Therefore, the boundary points of all market segments belong to Y. Q.E.D.

Given Lemma 7, the shortest path method described in §4.1 can be modified to provide the *exact* optimal solution to the assortment problem under a discrete distribution. Let $V = \{1, \ldots, N, N+1\}$ be a set of nodes in an acyclic network. Nodes $1, \ldots, N$ correspond to consumer locations $y_1, \ldots, y_N$, and node $N+1$ is a dummy node. Let $A = \{(v, v') : v, v' \in V \text{ and } v < v'\}$ be the set of arcs. We have $|A| = N(N+1)/2$. A unit flow on any arc $(v, v') \in A$ means that consumer locations $y_v, \ldots, y_{v'-1}$ are either covered by the same product, i.e., $[y_v, y_{v'-1}]$ is a market segment in the assortment, or constitute a portion of the product space that is not covered. For each arc $(v, v') \in A$, we define r as the profit contribution of the product if the locations are covered and zero otherwise:

$$r(v, v') = \max\left\{0, -k + (\bar{p} - T(y_{v'-1} - y_v) - c)\, m \sum_{j=v}^{v'-1} g_j^N\right\}.$$

As in §4.1, the complexity of this method is $O(N^2)$.

It is also possible to show that Theorem 2 continues to hold with the following definition of the likelihood ratio order:

Definition 3. Let F and G be two discrete distributions with the same set Y of consumer locations. Let θ_j and θ'_j be the probability of a consumer in location y_j under F and G respectively. G LR-dominates F, or $G \succeq_{LR} F$, if $\frac{\theta'_j}{\theta_j}$ is non-decreasing in j.

Appendix C: Remarks on Theorem 2

The following example shows that Theorem 2 does not necessarily hold under FOSD.

Example 4. Let $k=5$, $\bar{p}=20$, $c=5$, $d(z)=29|z|$ and $m=4$. Consider two piecewise constant density functions. F is such that:

$$F(x) = \begin{cases} 0.82/0.25x & \text{if } 0 \le x \le 0.25 \\ 0.82+0.10/(0.5-0.25)(x-0.25) & \text{if } 0.25 < x \le 0.5 \\ 0.92+0.08/(0.75-0.5)(x-0.5) & \text{if } 0.5 < x \le 0.75 \end{cases}$$

and G is such that:

$$G(x) = \begin{cases} 0.8/0.25x & \text{if } 0 \le x \le 0.25 \\ 0.80+0.12/(0.5-0.25)(x-0.25) & \text{if } 0.25 < x \le 0.5 \\ 0.92+0.08/(0.75-0.5)(x-0.5) & \text{if } 0.5 < x \le 0.75 \end{cases}$$

We have $G \succeq_{FOSD} F$ but not $G \succeq_{LR} F$. The optimal assortment for F is $\mathbf{a}^*$=([0, 0.125], [0.125,0.25], [0.25, 0.736]) and the optimal assortment for G is $\mathbf{a}'$=([0, 0.125], [0.125,0.25], [0.25, 0.7047]) and we have $\mathbf{a}^* \succeq_R \mathbf{a}'$.

Also note that the difference between the profits from two assortments ordered with respect to $\succeq_R$ is neither monotone nor single-crossing (see Milgrom and Shannon 1994) as the consumer taste distribution shifts to the right: $\pi(\mathbf{a}';F) \ge \pi(\mathbf{a};F)$ does not imply $\pi(\mathbf{a}';G) \ge \pi(\mathbf{a};G)$ for $\mathbf{a}'$ to the right of $\mathbf{a}$ and $G \succeq_{LR} F$. However, the optimal assortments satisfy a weaker property similar to Interval Dominance (see Strulovici and Quah 2009). Hence, the optimality of $\mathbf{a}^*$ for distribution $\boldsymbol{\theta}$ is a necessary condition in Theorem 2.